Convexified Gauss curvature flow of bounded open sets in an anisotropic external field: a stochastic approximation and PDE

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Abstract

We construct a random crystalline (or polyhedral) approximation of a convexified Gauss curvature flow of bounded open sets in an anisotropic external field. We also show that a weak solution to the PDE which describes the motion of a bounded open set is unique and is a viscosity solution of it.

1 Introduction

Gauss curvature flow is known as a mathematical model of the wearing process of a convex stone rolling on a beach and has been studied by many authors (see e.g. [2, 3, 6, 7, 11, 14, 24]).

In the last few years we have been generalizing the theory of Gauss curvature flow to a class of nonconvex sets.

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In [17] we studied the existence and the uniqueness of a viscosity solution to the PDE that describes the time evolution of a nonconvex graph by a convexified Gauss curvature (see (1.10) for PDE).

In [19] we proposed and studied the discrete stochastic approximations of evolving functions which are generalizations of those considered in [17], and proved the existence and the uniqueness of a weak solution to the PDE which appears in the continuum limit of discrete stochastic processes, and discussed under what conditions a weak solution to the PDE is a viscosity solution of it.

In [20] we studied the existence and the uniqueness of the motion (or time evolution) of a nonconvex compact set which evolves by a convexified Gauss curvature in $\mathbb{R}^N$ ($N \geq 2$), by the level set approach in the theory of viscosity solutions (see e.g. [5, 10, 23] for the level set approach).

We introduce the notion of the motion of a smooth oriented closed hypersurface by a convexified Gauss curvature.

Let $M$ be a smooth oriented closed hypersurface in $\mathbb{R}^N$ and $e$ be a smooth vector field over $M$ of unit normal vectors. For $x \in M$, let $T_x M$ denote the tangent space of $M$ at $x$, and let $A_x : T_x M \mapsto T_x M$ denote the Weingarten map at $x$ defined by the following:

$$A_x(v) = -D_v e \quad \text{for } v \in T_x M, \quad (1.1)$$

where $D_v e$ denotes the derivative of $e$ with respect to $v$. Recall that the principal curvatures $\kappa_1, \ldots, \kappa_n$ ($n := N - 1$) of $M$ at $x$ are the eigenvalues of the symmetric map $A_x$ and the Gauss curvature $K(x)$ of $M$ at $x$ is given by $\det A_x$.

Let $C$ be the convex hull $\text{co} M$ of $M$. We define $\sigma : M \mapsto \{0, 1\}$ by

$$\sigma(x) = \begin{cases} 1 & \text{if } x \in M \cap \partial C, \\ 0 & \text{otherwise,} \end{cases} \quad (1.2)$$

and call $\sigma(x)K(x)$ the convexified Gauss curvature of $M$ at $x$.

The motion of a smooth oriented closed hypersurface by a convexified Gauss curvature is the curvature flow:

$$v = -\sigma K \nu, \quad (1.3)$$

where $\nu$ denotes the unit outward normal vector on the surface and $v$ denotes the velocity of the surface.
Let \((A_x)_+\) denote the positive part of the symmetric map \(A_x\). \(K_+(x) := \det \{(A_x)_+\}\) is called the positive part of the Gauss curvature of \(M\) at \(x\), and the following holds:

\[
\sigma(x)K(x) = \sigma(x)K_+(x).
\]

(1.4)

**Remark 1** For \(x \in M\),

\[
\det \{(A_x)_+\} = \begin{cases} 
\det A_x & \text{if } A_x \text{ is nonnegative definite,} \\
0 & \text{otherwise.}
\end{cases}
\]

(1.5)

The crystalline (or polyhedral) approximation of a smooth simple closed convex curve which evolves as the curvature flow was considered by Girão and is useful in the numerical analysis (see [13] and the references therein). We refer to [12] and the references therein for the recent development of this topics.

When \(N = 2\), the discrete stochastic approximation of the curvature flow of smooth simple closed convex curves was given in [18] where the model and the approach are completely different from those in this paper.

In this paper we propose and study the discrete stochastic approximation of a convexified Gauss curvature flow of bounded open sets in an anisotropic external field. Our result in this paper is the first one in case \(N \geq 3\), among random and nonrandom results, which gives a crystalline approximation of the motion of a bounded open set in \(\mathbb{R}^N\) by Gauss curvature.

We briefly describe what we proved in [19], and then discuss the results in this paper more precisely to compare a convexified Gauss curvature flow of graphs with that of closed hypersurfaces.

For \(x \in \mathbb{R}^n\) and \(u : \mathbb{R}^n \mapsto \mathbb{R}\), the following set is called the subdifferential of \(u\) at \(x\):

\[
\partial u(x) := \{p \in \mathbb{R}^n : u(y) - u(x) \geq p \cdot (y - x) \text{ for all } y \in \mathbb{R}^n\},
\]

(1.6)

where \(\cdot\) denotes the inner product in \(\mathbb{R}^n\).

Alexandrov-Bakelman’s generalized curvature introduced in the following played a crucial role in [19].

**Definition 1** (see e.g. [4, section 9.6]). Let \(R \in L^1(\mathbb{R}^n : [0, \infty), dx)\) and \(u \in C(\mathbb{R}^n)\). For \(A \in B(\mathbb{R}^n)\) (:=Borel \(\sigma\)-field of \(\mathbb{R}^n\)), put
\[
  w(R, u, A) := \int_{\cup_{x \in A} \partial u(x)} R(y)dy \quad (A \in B(\mathbb{R}^n)).
\]  

Let \( T \in [0, \infty) \) and \( R \in L^1(\mathbb{R}^n ; [0, \infty), dx) \). We showed the existence and the uniqueness of a solution \( u \in C([0, T) \times \mathbb{R}^n) \) to the following equation (see [19, Theorem 1]): for any \( \varphi \in C_c(\mathbb{R}^n) \) and any \( t \in [0, T) \),

\[
  \int_{\mathbb{R}^n} \varphi(x)(u(t, x) - u(0, x))dx = \int_0^t ds \int_{\mathbb{R}^n} \varphi(x)w(R, u(s, \cdot), dx).
\]  

The existence of a continuous solution to (1.8) was given by the continuum limit of the infinite particle systems \( \{ (Z_m(t, z))_{z \in \mathbb{Z}^n/m} \}_{t \geq 0} \) that satisfies the following: for any \( t \geq 0 \) and any \( z \in \mathbb{Z}^n/m \),

\[
  P(Z_m(t + \Delta t, z) - Z_m(t, z) > 0) = m^n E[w(R, \hat{Z}_m(t, \cdot), \{ z \})] \Delta t + o(\Delta t)
\]  
as \( \Delta t \to 0 \) (\( m \geq 1 \)), where \( \hat{Z}_m(t, \cdot) \) denotes a convex envelope of the function \( z \mapsto Z_m(t, z) \), i.e., the graph of the boundary of the convex hull, in \( \mathbb{R}^N \), of the set \( \{(z, y) | z \in \mathbb{Z}^n/m, y \geq Z_m(t, z)\} \).

In [19, Theorem 2], we proved that a continuous solution \( u \) to (1.8) sweeps in time \( t > 0 \) a region with volume given by \( t \cdot w(R, u(0, \cdot), \mathbb{R}^n) \), and that, for continuous solutions \( u \) and \( v \) to (1.8) with \( v(0, \cdot) = \hat{u}(0, \cdot), \hat{u}(t, \cdot) \) is different from \( v(t, \cdot) \) at time \( t > 0 \) in general if \( u(0, \cdot) \neq \hat{u}(0, \cdot) \).

We also showed that a continuous solution to (1.8) is a viscosity solution of the following PDE (see [19, Theorem 3]):

\[
  \partial_t u(t, x) = \chi(u, Du(t, x), t, x)R(Du(t, x))\text{Det}_+(D^2 u(t, x))
\]  

\( ((t, x) \in (0, \infty) \times \mathbb{R}^n) \), where \( Du(t, x) := (\partial u(t, x)/\partial x_i)_{i=1}^n, D^2 u(t, x) := (\partial^2 u(t, x)/\partial x_i \partial x_j)_{i,j=1}^n \),

\[
  \chi(u, p, t, x) := \begin{cases} 
  1 & \text{if } p \in \partial u(t, x), \\
  0 & \text{otherwise}
\end{cases}
\]

(\( \partial u(t, x) \) denotes the subdifferential of the function \( x \mapsto u(t, x) \)). Conversely, we discussed under what conditions a viscosity solution to (1.10) is a solution to (1.8).
Remark 2 When \( R(p) = (1 + |p|^2)^{-(\alpha + 1)/2} \),

\[
(1 + |Du(t, x)|^2)^{-1/2} \chi(u, Du(t, x), t, x)R(Du(t, x))\text{Det}_+(D^2u(t, x))
\]

can be considered as the convexified Gauss curvature of \( \{(y, u(t, y))|y \in \mathbb{R}^N\} \) at \( x \) if we consider \( \{(y, z)|y \in \mathbb{R}^N, z \geq u(t, y)\} \) as the inside of the hypersurface \( \{(y, u(t, y))|y \in \mathbb{R}^N\} \).

Next we briefly discuss what we study in this paper.

Let \( F \) be a closed convex set in \( \mathbb{R}^N \). For \( x \in \partial F \), put

\[
N_F(x) := \{p \in S^{N-1} | F \subset \{y| < y - x, p > \leq 0\}\},
\]

where \( < \cdot, \cdot > \) denotes the inner product in \( \mathbb{R}^N \).

To consider a convexified Gauss curvature flow of bounded open sets by the level set approach, we introduce new types of measures.

Definition 2 Let \( u \) be a bounded function from a subset of \( \mathbb{R}^N \) to \( \mathbb{R} \), and \( R \in L^1(S^{N-1} : [0, \infty), d\mathcal{H}^{N-1}) \), where \( d\mathcal{H}^{N-1} \) denotes a \( (N - 1) \)-dimensional Hausdorff outer measure.

(i). Let \( r \in \mathbb{R} \). For \( B \in B(\mathbb{R}^N) \), put

\[
\omega_r(R, u, B) := \int_{N_{co u^{-1}([r, \infty))} \cap (\partial (co u^{-1}([r, \infty)) \cap B)} R(p)d\mathcal{H}^{N-1}(p), \tag{1.11}
\]

where \( A^- \) denotes the closure of the set \( A \).

(ii). For \( B \in B(\mathbb{R}^N) \), put

\[
\mathbf{w}(R, u, B) := \int_{\mathbb{R}} dr \omega_r(R, u, B), \tag{1.12}
\]

provided the right hand side is well defined.

When it is not confusing, we write \( \omega_r(R, u, dx) = \omega_r(u, dx) \) and \( \mathbf{w}(R, u, dx) = \mathbf{w}(u, dx) \) for the sake of simplicity.

The existence and the uniqueness of a solution to the following equation is given in section 2.

Definition 3 Let \( T \in [0, \infty] \) and \( R \in L^1(S^{N-1} : [0, \infty), d\mathcal{H}^{N-1}) \). A family of bounded open sets \( \{D(t)\}_{t \in [0, T]} \) in \( \mathbb{R}^N \) is called a convexified Gauss curvature flow in an \((R-)\)anisotropic external field on \([0, T)\) if
\[ D(t) = (\sigma D(t)) \cap D(0) \quad \text{for } t \in [0, T), \quad (1.13) \]

and if the following holds: for any \( \varphi \in C_0(\mathbb{R}^N) \) and any \( t \in [0, T), \)

\[
\int_{\mathbb{R}^N} \varphi(x)(I_{D(0)}(x) - I_{D(t)}(x)) dx = \int_0^t ds \int_{\mathbb{R}^N} \varphi(x) \omega(R, I_{D(s)}(\cdot), dx). \quad (1.14)
\]

We also show the existence and the uniqueness of a solution \( u \in C_0([0, T] \times \mathbb{R}^N) \) to the following: for any \( \varphi \in C_0(\mathbb{R}^N) \) and any \( t \in [0, T), \)

\[
\int_{\mathbb{R}^N} \varphi(x)(u(0, x) - u(t, x)) dx = \int_0^t ds \int_{\mathbb{R}^N} \varphi(x) w(R, u(s, \cdot), dx). \quad (1.15)
\]

The existence of \( \{I_{D(t)}\}_{t \geq 0} \) in Definition 3 is given by the continuum limit of a class of particle systems \( \{(Y_m(t, z))_{z \in \mathbb{Z}^N/m}\}_{t \geq 0} \) that satisfies the follows: for any \( t \geq 0 \) and any \( z \in \mathbb{Z}^N/m, \)

\[
P(Y_m(t + \Delta t, z) - Y_m(t, z) < 0) = m^N E[\omega(Y_m(t, \cdot), \{z\})] \Delta t + o(\Delta t) \quad (1.16)
\]
as \( \Delta t \to 0 \) (\( m \geq 1 \)) (see Theorem 1 in section 2).

The existence and the uniqueness of a solution to (1.15) will be given by the continuum limit of the linear combinations of solutions to (1.14) with \( D(0) = u(0, \cdot)^{-1}(\{r, \infty\}) \) for \( r \in \mathbb{R} \) (see Corollary 2 in section 2).

We also discuss the properties of \( \{D(t)\}_{t \geq 0} \) in Definition 3 (see Theorem 2 in section 2).

For \( p \in \mathbb{R}^N \) and a \( N \times N \)-symmetric real matrix \( X \), put

\[
G(p, X) := \begin{cases} 
|p| \det + \left( -\frac{1}{|p|} \tilde{p} \otimes \tilde{p} - (p \otimes \tilde{p}) + \tilde{p} \otimes \tilde{p} \right) & \text{if } p \neq 0, \\
0 & \text{if } p = 0 
\end{cases} \quad (1.17)
\]

(see (1.4) for notation), where \( \tilde{p} := p/|p| \).

Suppose that a smooth oriented hypersurface \( M \) in \( \mathbb{R}^N \) is given by \( M = \{y \in \mathbb{R}^N \mid \varphi(y) = a, \ D\varphi(y) \neq 0\} \) for some \( \varphi \in C^2(\mathbb{R}^N) \) and \( a \in \mathbb{R} \), and that the vector field \( e \) is given by \( e_x = D\varphi(x)/|D\varphi(x)|. \) Regard the tangent space, \( T_xM \), as the orthogonal complement of \( e_x \), and let \( E_x := \text{span } e_x \) and \( \text{id}_{E_x} \) denote the identity map on \( E_x \). Then the map
\[ A_x \oplus \text{id}_{E_x} : \mathbb{R}^N \equiv T_x M \oplus E_x \to T_x M \oplus E_x \]

has a matrix representation

\[-(I - \bar{p} \otimes \bar{p}) \frac{X}{|p|} (I - \bar{p} \otimes \bar{p}) + \bar{p} \otimes \bar{p},\]

with \( p = D\varphi(x) \) and \( X = D^2\varphi(x) \). Therefore,

\[
K(x) = \det \left( -(I - \bar{p} \otimes \bar{p}) \frac{X}{|p|} (I - \bar{p} \otimes \bar{p}) + \bar{p} \otimes \bar{p} \right), \quad (1.18)
\]

\[
K_+(x) = \frac{G(p, X)}{|p|}. \quad (1.19)
\]

For \( \{D(t)\}_{t \geq 0} \) in Definition 3, we show that \( I_{D(t)}(x) \) and \( I_{D(t)}^-(x) \) are respectively a viscosity supersolution and a viscosity subsolution of the following PDE (see Theorem 3 in section 2):

\[
\partial_t u(t, x) + R \left( \frac{Du(t, x)}{|Du(t, x)|} \right) \sigma^-(u, Du(t, x), t, x) G(Du(t, x), D^2u(t, x)) = 0
\]

\[((t, x) \in (0, \infty) \times \mathbb{R}^N)\). Here

\[
\sigma^-(u, p, t, x) := \begin{cases} 
1 & \text{if } u(t, \cdot) < u(t, x) \text{ on } H(p, x) \text{ and } p \in \mathbb{R}^N \setminus \{o\}, \\
0 & \text{otherwise},
\end{cases} \quad (1.21)
\]

where

\[
H(p, x) := \{ y \in \mathbb{R}^N \setminus \{x\} | < y - x, p > \leq 0 \}. \quad (1.22)
\]

Moreover, we show that a continuous solution to (1.15) is a viscosity solution of (1.20) (see Corollary 3 in section 2).

In [21], we will study the uniqueness of a viscosity solution to (1.20), from which we conclude that a viscosity solution to (1.20) with a bounded continuous initial data is a unique solution to (1.15).

Since \( G(p, X) \) is singular at \( p = o \), the standard definition of a viscosity solution (see [8]) is not appropriate for (1.20). We take the definition of a viscosity solution to (1.20) from [22].
We first introduce the set of admissible test functions. We denote by $\mathcal{F}$ the set of all functions $f \in C^2([0, \infty))$ for which $f'' > 0$ on $(0, \infty)$ and

$$\lim_{r \to 0} \frac{f(r)}{r^N} = 0.$$  \hfill (1.23)

Let $\Omega$ be an open subset of $(0, \infty) \times \mathbb{R}^N$. A function $\varphi \in C^2(\Omega)$ is called admissible in $\Omega$ if for any $(\hat{t}, \hat{x}) \in \Omega$ for which $D\varphi$ vanishes, there exists $f \in \mathcal{F}$ such that as $(t, x) \to (\hat{t}, \hat{x})$,

$$|\varphi(t, x) - \varphi(\hat{t}, \hat{x}) - \partial_t \varphi(\hat{t}, \hat{x})(t - \hat{t})| \leq f(|x - \hat{x}|) + o(|t - \hat{t}|).$$ \hfill (1.24)

We denote by $\mathcal{A}(\Omega)$ the set of all admissible functions in $\Omega$.

**Remark 3** $f(r) = r^{N+1} \in \mathcal{F}$ and $\varphi(t, x) = f(|x - \hat{x}|) \in \mathcal{A}((0, \infty) \times \mathbb{R}^N)$ for any $\hat{x} \in \mathbb{R}^N$.

**Definition 4 (Viscosity solution)** Let $0 < T \leq \infty$ and set $\Omega := (0, T) \times \mathbb{R}^N$, and put $R(\alpha/|\alpha|) := 0$.

(i). A function $u \in \text{LSC}(\Omega)$ is called a viscosity supersolution of (1.20) in $\Omega$ if whenever $\varphi \in \mathcal{A}(\Omega)$, $(s, y) \in \Omega$, and $u - \varphi$ attains a local minimum at $(s, y)$, then

$$\partial_t \varphi(s, y) + R\left(\frac{D\varphi(s, y)}{|D\varphi(s, y)|}\right) \sigma^+(u, D\varphi(s, y), s, y)G(D\varphi(s, y), D^2\varphi(s, y)) \geq 0,$$ \hfill (1.25)

where

$$\sigma^+(u, p, s, y) := \begin{cases} 1 & \text{if } u(s, \cdot) \leq u(s, y) \text{ on } H(p, y) \text{ and } p \in \mathbb{R}^N \setminus \{0\}, \\ 0 & \text{otherwise}. \end{cases}$$ \hfill (1.26)

(ii). A function $u \in \text{USC}(\Omega)$ is called a viscosity subsolution of (1.20) in $\Omega$ if whenever $\varphi \in \mathcal{A}(\Omega)$, $(s, y) \in \Omega$, and $u - \varphi$ attains a local maximum at $(s, y)$, then

$$\partial_t \varphi(s, y) + \sigma^-(u, D\varphi(s, y), s, y)R\left(\frac{D\varphi(s, y)}{|D\varphi(s, y)|}\right)G(D\varphi(s, y), D^2\varphi(s, y)) \leq 0.$$ \hfill (1.27)
(iii). A function $u \in C(\Omega)$ is called a viscosity solution of (1.20) in $\Omega$ if it is
a viscosity supersolution and a viscosity subsolution of (1.20) in $\Omega$.

**Remark 4** $\sigma^+(u, p, s, y) \geq \sigma^-(u, p, s, y)$ for all $u : \Omega \mapsto \mathbb{R}$ and all $(p, s, y) \in \mathbb{R}^N \times \Omega$.

Let $\mathcal{A}_0(\Omega)$ denote the set of all $\phi_1(t) + \phi_2(x) \in \mathcal{A}(\Omega)$ such that $x \mapsto G(D\phi_2(x), D^2\phi_2(x))$ is continuous in $\Omega$. Then one can replace, in Definition 5, $\mathcal{A}(\Omega)$ by $\mathcal{A}_0(\Omega)$ (see [20]).

**Remark 5** For any $f \in \mathcal{F}$ and $\hat{x} \in \mathbb{R}^N$, $\varphi(t, x) = f(|x - \hat{x}|) \in \mathcal{A}_0((0, \infty) \times \mathbb{R}^N)$.

In section 2 we state our main results which will be proved in section 4. In section 3 we give technical lemmas.
2 Main Result

In this section we give our main result.

We give two assumptions to state the stochastic process which approximates the solution to (1.13)-(1.14).

(A.0). $D$ is a bounded open set in $\mathbf{R}^N$.

(A.1). $R \in L^1(S^{N-1} : [0, \infty), d\mathcal{H}^{N-1})$ and $||R||_{L^1(S^{N-1})} = 1$.

Take $K > 0$ so that $\partial D \subset [-K + 1, K - 1]^N$. For $m \geq 1$, put

$$
\mathcal{S}_m := \{ I_A : [-K, K]^N \cap (\mathbf{Z}^N / m) \mapsto \{ 0, 1 \}| A \subset [-K, K]^N \cap \mathbf{Z}^N / m \},
$$

$$
D_m := D \cap (\mathbf{Z}^N / m).
$$

For $x, z \in \mathbf{Z}^N / m$ and $v \in \mathcal{S}_m$, put

$$
v_{m,z}(x) := \begin{cases} 
v(x) & \text{if } x \neq z, \\
0 & \text{if } x = z \end{cases}
$$

and for $f : \mathcal{S}_m \mapsto \mathbf{R}$, put

$$
A_m f(v) := m^N \sum_{z \in [-K, K]^N \cap (\mathbf{Z}^N / m)} \omega_1(R, v, \{ z \}) \{ f(v_{m,z}) - f(v) \}.
$$

Let $\{ Y_m(t, \cdot) \}_{t \geq 0}$ be a Markov process on $\mathcal{S}_m$ ($m \geq 1$), with the generator $A_m$, such that $Y_m(0, z) = I_{D_m}(z)$ ($z \in [-K, K]^N \cap (\mathbf{Z}^N / m)$). For $(t, x) \in [0, \infty) \times [-K, K]^N$, put also

$$
X_m(t, x) := \lim_{\tau \to \infty, \tau \cap D} I_{(\infty, \tau) \cap D}(x),
$$

where $A^c$ denotes the interior of the set $A \subset \mathbf{R}^N$.

Then $\{ X_m(t, \cdot) \}_{t \geq 0}$ is a stochastic process on

$$
\mathcal{S} := \{ f \in L^2([-K, K]^N) : ||f||_{L^2([-K, K]^N)} \leq (2K)^N \}
$$

which is a complete separable metric space by the metric

$$
d_\mathcal{S}(f, g) := \sum_{k=1}^{\infty} \max \{ \frac{||f - g, e_k >_{L^2([-K, K]^N)} ||}{2^k} \}.
$$

Here $\{ e_k \}_{k \geq 1}$ denotes a complete orthogonal basis of $L^2([-K, K]^N)$.

The following is our main result.
**Theorem 1** Suppose that (A.0)-(A.1) hold. Then there exists a unique solution \( \{D(t)\}_{t \geq 0} \) to (1.13)-(1.14) with \( D(0) = D \) on \([0, \infty)\) such that \( I_D(\cdot) \in C([0, \infty) : L^2([-K, K]^N)) \) and that the following holds: for any \( \gamma > 0 \),

\[
\lim_{m \to \infty} P(\sup_{t \geq 0} ||X_m(t, \cdot) - I_D(t)(\cdot)||_{L^2([-K, K]^N)} \geq \gamma) = 0. \tag{2.7}
\]

We recall Hausdorff metric of compact sets \( A \) and \( B \subset \mathbb{R}^N \):

\[
d_H(A, B) := \max(\max_{p \in A} \text{dist}(p, B), \max_{q \in B} \text{dist}(q, A)). \tag{2.8}
\]

As a corollary, we obtain

**Corollary 1** Suppose that (A.0)-(A.1) hold and that \( D \) is convex. Then for a unique solution \( \{D(t)\}_{t \geq 0} \) to (1.13)-(1.14) with \( D(0) = D \) on \([0, \infty)\), the following holds: for any \( T \in [0, \text{Vol}(D)) \) and any \( \gamma > 0 \),

\[
\lim_{m \to \infty} P(\sup_{0 \leq t \leq T} d_H(\text{dist}(Y_m(t, \cdot)^{-1}(1), D(t)) \geq \gamma) = 0. \tag{2.9}
\]

We introduce the assumption on the initial function in the equation (1.15).

(A.2). \( h \in C_b(\mathbb{R}^N) \). For any \( r \in \mathbb{R}^N \), the set \( h^{-1}((r, \infty)) \) is bounded or \( \mathbb{R}^N \).

Then one can easily obtain the following from Theorem 1.

**Corollary 2** Suppose that (A.1)-(A.2) hold. Then there exists a unique continuous solution \( \{u(t, \cdot)\}_{t \geq 0} \) to (1.15) with \( u(0, \cdot) = h(\cdot) \) on \([0, \infty)\). In addition, for any \( r \in \mathbb{R} \), \( \{u(t, \cdot)^{-1}((r, \infty))\}_{t \geq 0} \) is a unique solution to (1.13)-(1.14) with \( D(0) = h^{-1}((r, \infty)) \) on \([0, \infty)\).

The following theorem collects some of elementary properties of solutions to (1.13)-(1.14).

**Theorem 2** Suppose that (A.0)-(A.1) hold. Let \( \{D(t)\}_{t \geq 0} \) be a unique solution to (1.13)-(1.14) with \( D(0) = D \) on \([0, \infty)\). Then the following holds.

(a) \( t \mapsto D(t) \) is nonincreasing on \([0, \infty)\).
(b) For any \( t \leq T^* := \text{Vol}(D(0)) \),

\[
\text{Vol}(D(0) \setminus D(t)) = t. \tag{2.10}
\]
(c) $D(t) = \emptyset$ for $t \geq T^*$.

(d) Let $\{D_1(t)\}_{t \geq 0}$ be a solution to (1.13)-(1.14) on $[0, \infty)$ such that $D_1(0)$ is a bounded, convex, open set which contains $D$. Then

$$D(t) \subset D_1(t) \quad \text{for all } t \geq 0,$$

(2.11)

where the equality holds if and only if $D(0) = D_1(0)$.

Under

(A.3). $R \in C(S^{N-1} : [0, \infty))$,

we give the relation between the solution to (1.13)-(1.14) and the viscosity solution of (1.20).

**Theorem 3** Suppose that (A.0)-(A.1) and (A.3) hold. Then for a unique solution $\{D(t)\}_{t \geq 0}$ to (1.13)-(1.14) with $D(0) = D$ on $[0, \infty)$, $I_{D(t)}(x)$ and $I_{D(t)}^{-}(x)$ is a viscosity supersolution and a viscosity subsolution to (1.20) in $(0, \infty) \times \mathbb{R}^N$, respectively.

As a corollary, we obtain

**Corollary 3** Suppose that (A.1)-(A.3) hold. Then a solution $\{u(t, \cdot)\}_{t \geq 0}$ to (1.15) with $u(0, \cdot) = h(\cdot)$ on $[0, \infty)$ is a viscosity solution to (1.20) in $(0, \infty) \times \mathbb{R}^N$. 
3 Lemma

In this section we give lemmas which will be used in the next section. We extend $Y_m(t, \cdot)$ as a function on $\mathbb{R}^N$ so that

$$Y_m(t, x) = \begin{cases} 0 & (x \in D^c \cap (\mathbb{Z}^N/m)), \\ Y_m(t, [mx]/m) & (x = (x_i)_{i=1}^N \in \mathbb{R}^N), \end{cases} \tag{3.1}$$

where $[mx] := ([mx_i])_{i=1}^N$ and $[mx_i]$ denotes an integer part of $mx_i$.

Remark 6 For $z \in \mathbb{Z}^N/m$,

$$Y_m(t, z) = \frac{1}{m^N} \int_{x \in \mathbb{R}^N \mid [mx] = mz} Y_m(t, x) dx.$$

Lemma 1 Suppose that (A.0)-(A.1) hold. Then $\{Y_m(\cdot, \cdot)\}_{m \geq 1}$ is tight in $D([0, \infty) : \mathcal{S})$, and any weak limit point of $\{Y_m(\cdot, \cdot)\}_{m \geq 1}$ belongs to the set $C([0, \infty) : \mathcal{S})$.

(Proof). Since $\mathcal{S}$ is compact and since $t \mapsto Y_m(t, x)$ is nonincreasing for any $x \in \mathbb{R}^N$, we only have to show the following (see [9, p. 129, Corollary 7.4 and p. 148, Theorem 10.2]): for any $\eta > 0$ and $T > 0$, there exists $\delta > 0$ such that for any $i$ for which $1 \leq i \leq [T/\delta] + 1$,

$$\lim_{m \to \infty} P(||Y_m(i\delta, \cdot) - Y_m((i - 1)\delta, \cdot)||_{L^1([-K,K]^N)} \geq \eta) = 0. \tag{3.2}$$

Indeed, for any $s$ and $t$ for which $(i - 1)\delta \leq s \leq t \leq i\delta$,

$$Y_m(s, x) - Y_m(t, x) = 0 \text{ or } 1,$$

and

$$d_\mathcal{S}(Y_m(t, \cdot), Y_m(s, \cdot))^2 \leq ||Y_m(t, \cdot) - Y_m(s, \cdot)||^2_{L^2([-K,K]^N)} = ||Y_m(i\delta, \cdot) - Y_m((i - 1)\delta, \cdot)||^2_{L^2([-K,K]^N)}.$$

For $\delta < \eta/2$ and $m \geq 1$, by Chebychev’s inequality and Itô’s formula (see [15]),

\[13\]
\[
P(||Y_m(i\delta, \cdot) - Y_m((i - 1)\delta, \cdot)||_{L^1([-K, K]^N)} \geq \eta) \leq \frac{2}{\eta} E[\sum_{z \in D_m} (Y_m(i\delta, z) - Y_m((i - 1)\delta, z)) \frac{1}{mN} \int_{(i-1)\delta}^{i\delta} \omega_1(Y_m(s, \cdot), D_m)ds]^2] \leq \frac{2}{\eta} m^{-N} \int_{(i-1)\delta}^{i\delta} \omega_1(Y_m(s, \cdot), D_m)ds]
\]
(see (2.2) for notation). Indeed,
\[
||Y_m(i\delta, \cdot) - Y_m((i - 1)\delta, \cdot)||_{L^1([-K, K]^N)} = - \sum_{z \in D_m} (Y_m(i\delta, z) - Y_m((i - 1)\delta, z)) \frac{1}{mN} \int_{(i-1)\delta}^{i\delta} \omega_1(Y_m(s, \cdot), D_m)ds + \int_{(i-1)\delta}^{i\delta} \omega_1(Y_m(s, \cdot), D_m)ds.
\]
Q. E. D.

**Remark 7** In (3.3), if \( Y_m(s, \cdot) \equiv 0 \), then \( \omega_1(Y_m(s, \cdot), D_m) = 0 \).

**Lemma 2** Suppose that (A.0)-(A.1) hold. Then there exist a subsequence \( \{m_k\}_{k \geq 1} \subset \mathbb{N} \) and stochastic processes \{\( Y_{1,m_k}(\cdot, \cdot) \)\}_{k \geq 1} on a probability space \((\Omega_1, \mathcal{B}_1, P_1)\) such that the probability law of \{\( Y_{1,m_k}(\cdot, \cdot) \)\}_{k \geq 1} is the same as that of \{\( Y_{m_k}(\cdot, \cdot) \)\}_{k \geq 1}, and such that \{\( Y_{1,m_k}(\cdot, \cdot) \)\}_{k \geq 1} is convergent in \( D([0, \infty) : \mathcal{S}) \), \( P_1 \)-almost surely, and such that the following holds \( P_1 \)-almost surely: for any \( T > 0 \) and \( \varphi \in C([-K, K]^N) \)
\[
\sup_{0 \leq t \leq T} | \sum_{z \in D_{m_k}} \varphi(z)(Y_{1,m_k}(t, z) - Y_{1,m_k}(0, z)) \frac{1}{m_k N} + \int_0^t \sum_{z \in D_{m_k}} \varphi(z)\omega_1(Y_{1,m_k}(s, \cdot), \{z\})ds | \to 0 \quad \text{as} \quad k \to \infty.
\]

Here \( Y_{1,m_k} \) is defined by \( Y_{1,m_k} \) in the same way as in Remark 6.
(Proof). By Lemma 1 and Skorohod’s Theorem (see [9, p. 102, Theorem 1.8]), there exist a subsequence \( \{m_{0,k}\}_{k \geq 1} \subset \mathbb{N} \) and stochastic processes \( \{\bar{Y}_{1,m_{0,k}}(\cdot, \cdot)\}_{k \geq 1} \) on a probability space \((\Omega_1, \mathcal{B}_1, P_1)\) such that the probability law of \( \{\bar{Y}_{1,m_{0,k}}(\cdot, \cdot)\}_{k \geq 1} \) is the same as that of \( \{\bar{Y}_{m_{0,k}}(\cdot, \cdot)\}_{k \geq 1} \), and such that \( \{\bar{Y}_{1,m_{0,k}}(\cdot, \cdot)\}_{k \geq 1} \) is convergent in \( D([0, \infty); S) \), \( P_1 \)-almost surely.

As in (3.3), by Doob-Kolmogorov’s inequality (see [15]), for any \( T > 0 \) and \( \varphi \in C([-K, K]^N) \)

\[
E_1 \left[ \sup_{0 \leq t \leq T} \left| \sum_{z \in \mathcal{D}_{m_{0,k}}} \varphi(z)(Y_{1,m_{0,k}}(t, z) - Y_{1,m_{0,k}}(0, z)) \frac{1}{(m_{0,k})^N} \right|^2 \right] + \int_0^T \sum_{z \in \mathcal{D}_{m_{0,k}}} \varphi(z) \omega_1(Y_{1,m_{0,k}}(s, \cdot), \{z\}) ds \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{3.5}
\]

Since a \( L^2 \)-convergent sequence of random variables has an almost surely-convergent subsequence, and since \( C([-K, K]^N) \) is separable, one can complete the proof by the diagonal method.

Q. E. D.

When it is not confusing, we write \( Y_{1,m_k} = Y_{m_k} \) and \( Y_{1,m_k} = Y_{m_k} \) on \((\Omega_1, \mathcal{B}_1, P_1)\) for the sake of simplicity.

Take \( x_0 \in D \) and \( r_0 > 0 \) so that \( U_{4r_0}(x_0) := \{y \in \mathbb{R}^N : |x_0 - y| < 4r_0\} \subset D \), and put \( U_0 := U_{2r_0}(x_0) \). Then

\[
V_0 := \inf_{x \in \partial U_0} \text{Vol}(U_{3r_0}(x_0) \cap H(x_0 - x, x)) > 0. \tag{3.6}
\]

Put, on \((\Omega_1, \mathcal{B}_1, P_1)\),

\[
\tau_m := \inf\{t > 0 | Y_{1,m}(t, z) = 0 \text{ for some } z \in (\mathbb{Z}^N/m) \cap U_0\}. \tag{3.7}
\]

Then the following holds.

**Lemma 3** Suppose that (A.0)-(A.1) hold. Then

\[
P_1(V_0 \leq \liminf_{k \rightarrow \infty} \tau_{m_k} \leq \limsup_{k \rightarrow \infty} \tau_{m_k} \leq \text{Vol}(D)) = 1. \tag{3.8}
\]
(Proof). By (3.4), for any $t > \text{Vol}(D)$,

$$
\limsup_{k \to \infty} \{ \min(\tau_{m_k}, t) \} \\
= \limsup_{k \to \infty} \int_0^{\min(\tau_{m_k}, t)} \omega_1(Y_{m_k}(s, \cdot), D_{m_k}) \, ds \\
\leq \limsup_{k \to \infty} \sum_{z \in D_{m_k}} (Y_{m_k}(0, z) - Y_{m_k}(\min(\tau_{m_k}, t), z)) \frac{1}{m_k^2} \leq \text{Vol}(D)
$$

$P_1$- almost surely. We also have

$$
V_0 \leq \liminf_{k \to \infty} \sum_{z \in D_{m_k}} (Y_{m_k}(0, z) - Y_{m_k}(\tau_{m_k}, z)) \frac{1}{m_k^2} \\
\leq \liminf_{k \to \infty} \int_{\tau_{m_k}}^{\tau_{m_k}} \omega_1(Y_{m_k}(s, \cdot), D_{m_k}) \, ds = \liminf_{k \to \infty} \tau_{m_k}
$$

$P_1$- almost surely. 

Q. E. D.

The following lemma can be proved in the same way as in [4, section 5.2] and the proof is omitted.

**Lemma 4** Suppose that (A.1) holds. Let $F$ and $F_m(m \geq 1)$ be closed convex sets in $\mathbb{R}^N$ such that $\partial F$ and $\partial F_m(m \geq 1)$ are closed hypersurfaces and such that $d_H(F_m, F) \to 0$ as $m \to \infty$. Then $\omega_1(I_{F_m}(\cdot), dx)$ weakly converges to $\omega_1(F, dx)$ as $m \to \infty$, that is, for any $\varphi \in C_0(\mathbb{R}^N),$

$$
\lim_{m \to \infty} \int_{\mathbb{R}^N} \varphi(x) \omega_1(I_{F_m}(\cdot), dx) = \int_{\mathbb{R}^N} \varphi(x) \omega_1(I_F(\cdot), dx). 
$$

We denote by $X(\cdot, \cdot) \in C([0, \infty) : \mathcal{S})$ the $P_1$-a.s. limit of $\mathbf{Y}_{1,m_k}(\cdot, \cdot)$ as $k \to \infty$. Then we have

**Lemma 5** Suppose that (A.0)-(A.1) hold. Then there exists a solution \{$D(t)\}_{t \in [0, V_0]}$ to (1.13)-(1.14) on $[0, V_0)$ such that the following holds $P_1$-almost surely:

$$
X(t, x) = I_{D(t)}(x), \quad dx - a.e. \quad \text{for all } t \in [0, V_0). 
$$

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(Proof). For $p \in S^{N-1}$, let $C(x_0, r_0; p)$ denote a semi-infinite cylinder

$$\{x_0 + rp + x : r \geq 0, |x| \leq r_0, <x, p> = 0, x \in \mathbb{R}^N\}$$

which can be obtained by moving a $(N - 1)$-dimensional ball

$$\{x_0 + x : |x| \leq r_0, <x, p> = 0, x \in \mathbb{R}^N\}$$
in the positive direction of $p$.

Take $p_1, \cdots, p_{k_0} \in S^{N-1}$ for some $k_0 \in \mathbb{N}$ so that

$$\operatorname{co} D \subset \bigcup_{i=1}^{k_0} C(x_0, r_0; p_i).$$

For $i = 1, \cdots, k_0$, take $\{q_{i1}, \cdots, q_{i(N-1)}\}$ so that $\{q_{i1}, \cdots, q_{i(N-1)}, p_i\}$ is an orthonormal basis in $\mathbb{R}^N$, and put

$$C_{m_k}(t) := \operatorname{co} Y_{m_k}(t, \cdot)^{-1}(1).$$  \hfill (3.13)

For $x = (x_k)_{k=1}^{N-1} \in \mathbb{R}^{N-1}$ for which $|x| \leq 2r_0$, put also

$$\tilde{X}_{m_k,i}(t, x) := -\sup\{r > 0 |x_0 + rp_i + \sum_{j=1}^{N-1} q_{ij} x_j \in C_{m_k}(t)\}. \hfill (3.14)$$

Then $\tilde{X}_{m_k,i}(t, \cdot)$ is a bounded convex function on $\{x \in \mathbb{R}^{N-1} : |x| \leq 7r_0/4\}$ for $t \in [0, \tau_{m_k})$ if $m_k \geq 8N^{1/2}/r_0$.

It is known that the set of bounded convex functions with the same domain is compact as the set of continuous functions defined on $K$ for every compact subset $K$ of the interior of their domain (see [4, section 3.3]).

Therefore, by Lemma 3 and the diagonal method, there exists a subsequence $\{\tilde{X}_{m,k,i}(t, \cdot)\}_{k \geq 1} \subset \{\tilde{X}_{m,k,i}(t, \cdot)\}_{k \geq 1}$ and a convex function $\tilde{X}_i(t, \cdot)$ such that for any $t \in Q \cap [0, V_0)$ and $i = 1, \cdots, k_0$,

$$\lim_{k \to \infty} \sup_{x \in \mathbb{R}^{N-1}, |x| \leq 3r_0/2} |\tilde{X}_{m_k,i}(t, x) - \tilde{X}_i(t, x)| = 0 \hfill (3.15)$$

(Notice that $\{m_k\}_{k \geq 1}$ can be random).

It is clear that there exists a nonincreasing family of compact convex sets $\{\tilde{C}(t)\}_{t \in Q \cap [0, V_0)}$ such that for any $t \in Q \cap [0, V_0)$,
\[
\lim_{k \to \infty} d_H(C_{\hat{m}_k}(t), \tilde{C}(t)) = 0, \quad (3.16)
\]
\[
X_i(t, x) = -\sup\{r > 0 | x_0 + r p_i + \sum_{j=1}^{N-1} q_{ij} x_j \in \tilde{C}(t)\}
\]
for all \(i = 1, \ldots, k_0\), and \(x = (x_k)_{k=1}^{N-1} \in \mathbb{R}^{N-1}\) for which \(|x| \leq 3r_0/2\).
In particular,
\[
\lim_{k \to \infty} \|X_{1,\hat{m}_k}(t, \cdot) - I_{\tilde{C}(t) \cap D}(\cdot)\|_{L^2([-K, K]^{N})} = 0 \quad (3.17)
\]
for all \(t \in \mathbb{Q} \cap [0, V_0)\), where \(X_{1,\hat{m}_k}\) is defined by \(Y_{1,\hat{m}_k}\) in the same way as in (2.4). When it is not confusing, we write \(X_{1,\hat{m}_k} = X_{\hat{m}_k}\) on \((\Omega_l, \mathcal{B}_1, P_l)\) for the sake of simplicity.

The following also holds: for all \(t \in [0, V_0) \cap \mathbb{Q}\),
\[
\lim_{k \to \infty} \|Y_{\hat{m}_k}(t, \cdot) - X_{\hat{m}_k}(t, \cdot)\|_{L^2([-K, K]^{N})} = 0. \quad (3.18)
\]
Indeed, if \(X_m(t, x) \neq Y_m(t, x)\), then
\[
\text{dist}(x, \partial(C_m(t)^o \cap D)) \leq \frac{N^{1/2}}{m}
\]
; and by (3.16), the volume of the \(N^{1/2}/\hat{m}_k\)-neighborhood of the set \(\partial D \cup \partial C_{\hat{m}_k}(t)\) converges to zero as \(k \to \infty\) for \(t \in [0, V_0) \cap \mathbb{Q}\).

For \(t \in [0, V_0) \setminus \mathbb{Q}\), put
\[
\tilde{C}(t) := \cap_{s \in \mathbb{Q} \cap [0, t]} \tilde{C}(s). \quad (3.19)
\]
Then, by (3.17)-(3.19), the following holds \(P_1\)-a.s.:
\[
X(t, x) = I_{\tilde{C}(t) \cap D}(x), \quad dx - a.e. \text{ for all } t \in [0, V_0), \quad (3.20)
\]
since \(\{Y_{\hat{m}_k}\}_{k \geq 1}\) is a subsequence of a convergent sequence \(\{Y_{m_k}\}_{k \geq 1}\) and since \(X \in C([0, \infty) : \mathcal{S})\) is the \(P_1\)-a.s. limit, in \(D([0, \infty) : \mathcal{S})\), of \(\{Y_{m_k}\}\) as \(k \to \infty\), and since \(\{\tilde{C}(t)\}_{t \in [0, V_0) \cap \mathbb{Q}}\) is nonincreasing in \(t\).

Put
\[ D(t) := \overset{C(t)}{\mathcal{O}} \cap D. \]  
(3.21)

Then (1.13) holds for all \( t \in [0, V_0) \), since \( D = D(0) \) by (3.17) and since

\[ D(t) \supset \{ \co(\overset{C(t)}{\mathcal{O}} \cap D) \} \cap D = (\co D(t)) \cap D \supset D(t) \cap D = D(t). \]

On \( [0, V_0) \),

\[ \omega_1(I_{\overset{C(t)}{\mathcal{O}}}(\cdot), dx) = \omega_1(I_{D(t)}(\cdot), dx) \quad dt - a.e., \]  
(3.22)

since

\[ \overset{C(t)}{\mathcal{O}} \setminus (\co D(t))^{-} \subset \overset{C(t)}{\mathcal{O}} \setminus D(t)^{-} \subset \overset{D}{\mathcal{O}} \]
by (3.21), where \( \overset{D}{\mathcal{O}} \) denotes a complement of \( D \), and since

\[ \int_0^{V_0} ds \omega_1(I_{\overset{C(t)}{\mathcal{O}}}(\cdot), D) = \int_{D^c} (I_{D(0)}(x) - I_{D(V_0)}(x)) dx = 0 \]

by (3.4), (3.20) and Lemma 4. Here we used the fact that (3.16) holds except for at most countably many \( t \in [0, V_0) \).

Indeed, \( t \mapsto C_{\overset{\mathcal{O}}{\mathcal{O}}}(t) \) is nonincreasing and (3.16) holds for all \( t \in \mathbb{Q} \cap [0, V_0) \). Therefore, if \( C_{\overset{\mathcal{O}}{\mathcal{O}}}(t) \) does not converge to \( \overset{C}{\mathcal{O}}(t) \) as \( k \to \infty \), then \( (\overset{C}{\mathcal{O}}(t) \setminus \overset{C(t+)}{\mathcal{O}})^o \) is not empty and has a positive Lebesgue measure by (3.19), where \( \overset{C(t+)}{\mathcal{O}} := \cup_{s > t} \overset{C}{\mathcal{O}}(s) \). Besides, \( (\overset{C(t)}{\mathcal{O}} \setminus \overset{C(t+)}{\mathcal{O}})^o \) are disjoint for different \( t \).

By (3.4), Lemma 4, (3.20)-(3.22), (1.14) holds for all \( t \in [0, V_0) \) since (3.16) holds except for at most countably many \( t \in [0, V_0) \) as we mentioned above.

Q. E. D.

The following lemma implies the uniqueness of the solution to (1.13)-(1.14).

**Lemma 6** Suppose that (A.1) hold. For \( T > 0 \), if \( \{D_i(t)\}_{0 \leq t < T} \ (i = 1, 2) \) are solutions to (1.13)-(1.14) on \([0, T)\) for which \( D_1(0) \subset D_2(0) \), then \( D_1(t) \subset D_2(t) \) for all \( t \in [0, T) \). In particular, for all \( t \in [0, \min(\text{Vol}(D_1(0)), T)) \),

\[ d_H(D_1(t), D_2(t)^c) \geq d_H(D_1(0), D_2(0)^c). \]  
(3.23)
(Proof). For each $t \geq 0$, put

\[ \tilde{D}(t) := D_1(t)^- \cap D_2(t)^c, u_i(t, \cdot) := I_{D_i(t)}(\cdot), \quad u_i^-(t, \cdot) := I_{D_i(t)^-}(\cdot). \]

\[ N_i(t) := \bigcup_{x \in \partial D(t) \cap \partial D_i(t)} \{ p \in S^{N-1} | \sigma^+(u_i, -p, t, x) = 1 \} \]

$(i = 1, 2)$. Then $N_2(t) \subset N_1(t)$.

Take a nondecreasing sequence $\{ \eta_n \}_{n \geq 1}$ of nondecreasing $C^1$-functions such that

\[ \eta_n(r) = 0 \quad \text{for all } r \leq 0, \quad \eta_n(r) = 1 \quad \text{for all } r \geq \frac{1}{n}, \quad (3.24) \]

and for $r \in \mathbb{R}$, put

\[ \zeta_n(r) = \int_0^r \eta_n(s)ds. \quad (3.25) \]

Then since $t \mapsto u_i(t, x)$ and $t \mapsto u_i^-(t, x)$ are respectively right and left continuous for any $x \in \mathbb{R}^N$, for $t < \min(\text{Vol}(D_1(0)), T)$ and $x \in \mathbb{R}^N$,

\[ \zeta_n(u_i^-(t, x) - u_2(t, x) - 1) - \zeta_n(u_i^-(0, x) - u_2(0, x)) \]

\[ = \int_0^t \zeta_n(u_i^-(s, x) - u_2(s, x) - s/t)(u_i^-(ds, x) - u_2(ds, x)) \]

\[ - \frac{1}{t} \int_0^t \eta_n(u_i^-(s, x) - u_2(s, x) - s/t)ds. \]

Since $\zeta_n \geq 0$ and $D_1(0) \subset D_2(0)$, we have

\[ 0 \leq \int_0^t ds \int_{\mathbb{R}^N} \zeta_n(u_i^-(s, x) - u_2(s, x) - s/t) \]

\[ \times (\omega_1(u_2(s, \cdot), dx) - \omega_1(u_1(s, \cdot), dx)) \]

\[ - \frac{1}{t} \int_0^t ds \int_{\mathbb{R}^N} \eta_n(u_i^-(s, x) - u_2(s, x) - s/t)dx \]

\[ \rightarrow \int_0^t (1 - s/t)(\omega_1(u_2(s, \cdot), \tilde{D}(s)) - \omega_1(u_1(s, \cdot), \tilde{D}(s)))ds \]

\[ - \frac{1}{t} \int_0^t ds \int_{\tilde{D}(s)} dx \quad (\text{as } n \to \infty) \]

\[ \leq - \frac{1}{t} \int_0^t ds \int_{\tilde{D}(s)} dx, \]

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which implies the first assertion of this lemma.

Suppose that (3.23) dose not hold. Then there exists \( a \in (0, d_H(D_1(0), D_2(0)^c)) \) such that

\[
\inf\{d_H(D_1(t), D_2(t)^c)|t \in [0, \min(\text{Vol}(D_1(0)), T))\} < a.
\]

Take \( p_a \in S^{N-1} \) and \( t_a \in [0, \min(\text{Vol}(D_1(0)), T)) \) so that

\[
ap_a + D_1(t_a) \not\subset D_2(t_a).
\]

Since \( ap_a + D_1(0) \subset D_2(0) \) and \( \{ap_a + D_1(t)\}_{0 \leq t < T} \) is a solution to (1.13)-(1.14) on \([0, T)\), this contradicts the first assertion of this lemma.

Q. E. D.

Take \( \varphi \in C^2(\mathbb{R}^N) \) for which \( D\varphi(x_o) \neq 0 \) for some \( x_o \in \mathbb{R}^N \).

Let \( I_N \) denote a \( N \times N \)-identity matrix and put

\[
f_N := \frac{D\varphi(x_o)}{|D\varphi(x_o)|}, \quad (g_1 \cdots g_N) := I_N - f_N \otimes f_N.
\]

Take \( \{f_1, \cdots, f_{N-1}\} \) so that \( \{f_1, \cdots, f_N\} \) is an orthonormal basis of \( \mathbb{R}^N \). Then the following holds.

**Lemma 7**

(i) \( <g_i, f_N> = 0 \) \( (1 \leq i \leq N) \).

(ii) For \( i \) for which \( \partial_i \varphi(x_o) := \partial \varphi(x_o)/\partial x_i \neq 0 \),

\[
g_i = -\sum_{k \neq i} \frac{\partial_k \varphi(x_o)}{\partial \varphi(x_o)} g_k.
\]

(iii) \( \text{span}(g_1, \cdots, g_N) = \text{span}(f_1, \cdots, f_{N-1}) \).

(iv) \( D(D\varphi(x_o)/|D\varphi(x_o)|) \subset \text{span}(g_1, \cdots, g_N) \). As a mapping on \( \text{span}(g_1, \cdots, g_N) \), eigenvalues and eigenvectors of \( D(D\varphi(x_o)/|D\varphi(x_o)|) \) are the same as those of \( (g_1 \cdots g_N)(D^2\varphi(x_o)/|D\varphi(x_o)|)(g_1 \cdots g_N) \). In particular, all eigenvalues of \( D(D\varphi(x_o)/|D\varphi(x_o)|) \) are real.

(v) If eigenvalues \( \lambda_1 \leq \cdots \leq \lambda_{N-1} \) of \( -D(D\varphi(x_o)/|D\varphi(x_o)|) \) as a mapping on \( \text{span}(g_1, \cdots, g_N) \) are nonnegative, then

\[
\prod_{i=1}^{N-1} \lambda_i = \frac{G(D\varphi(x_o), D^2\varphi(x_o))}{|D\varphi(x_o)|}.
\]
(Proof). It is easy to see that (i) and (ii) hold. Take \( i \) for which \( \partial_i \varphi(x_o) \neq 0 \). Then, by (i) and (ii), we only have to show, to prove (iii), that \( \{g_j\}_{j \neq i} \) is independent. Suppose that for \( j = 1, \cdots, N \),

\[
\sum_{k \neq i} \lambda_k \left( \delta_{kj} - \frac{\partial_k \varphi(x_o) \partial_j \varphi(x_o)}{|D \varphi(x_o)|^2} \right) = 0. \tag{3.29}
\]

Putting \( j = i \) in (3.29), we obtain

\[
\sum_{k \neq i} \lambda_k \frac{\partial_k \varphi(x_o) \partial_i \varphi(x_o)}{|D \varphi(x_o)|^2} = 0,
\]

from which

\[
\sum_{k \neq i} \lambda_k \partial_k \varphi(x_o) = 0. \tag{3.30}
\]

Putting \( j \neq i \) in (3.29), we obtain

\[
\lambda_j - \partial_j \varphi(x_o) \sum_{k \neq i} \lambda_k \frac{\partial_k \varphi(x_o)}{|D \varphi(x_o)|^2} = 0,
\]

from which \( \lambda_j = 0 \) for \( j \neq i \), by (3.30).

We prove (iv). It is easy to see that

\[
D \left( \frac{D \varphi(x_o)}{|D \varphi(x_o)|} \right) = (g_1 \cdots g_N) \frac{D^2 \varphi(x_o)}{|D \varphi(x_o)|}. \tag{3.31}
\]

Hence

\[
D \left( \frac{D \varphi(x_o)}{|D \varphi(x_o)|} \right) (\sum_{i=1}^N x_i g_i) = \lambda \sum_{i=1}^N x_i g_i
\]

if and only if

\[
(g_1 \cdots g_N) \frac{D^2 \varphi(x_o)}{|D \varphi(x_o)|} (g_1 \cdots g_N) (\sum_{i=1}^N x_i g_i) = \lambda \sum_{i=1}^N x_i g_i,
\]

since

\[
(g_1 \cdots g_N)^2 = (g_1 \cdots g_N). \tag{3.32}
\]

Put \( P := (f_1 \cdots f_N) \) and \( Q := (f_1 \cdots f_{N-1}) \). The proof of (v) is divided into the following.
(STEP I) The eigenvalues of
\[-(I_N - f_N \otimes f_N)^2 \frac{D^2 \varphi(x_o)}{D\varphi(x_o)} (I_N - f_N \otimes f_N) + f_N \otimes f_N\]
are those of
\[
\begin{pmatrix}
-Q^* D \frac{D\varphi(x_o)}{D\varphi(x_o)} Q & o \\
\rho^* & 1
\end{pmatrix}.
\]

(STEP II) The eigenvalues of \(Q^* D (D\varphi(x_o) \langle D\varphi(x_o) \rangle Q\) are those of \(D(D\varphi(x_o) \langle D\varphi(x_o) \rangle \) on span \((g_1, \cdots, g_N)\).

(Proof of Step I). For \(\lambda \in \mathbb{R}\), denoting by \(P^*\) the transposed matrix of \(P\),

\[
det \left( -(I_N - f_N \otimes f_N)^2 \frac{D^2 \varphi(x_o)}{D\varphi(x_o)} (I_N - f_N \otimes f_N) + f_N \otimes f_N - \lambda I_N \right)
= \det \left( \left( I_{N-1} \ o^* \ o \right) P^* D^2 \varphi(x_o) D\varphi(x_o) P \left( I_{N-1} \ o^* \ o \right) + \left( O \ o^* \ 1 \right) - \lambda I_N \right)
= \det \left( \left( -Q^* \frac{D^2 \varphi(x_o)}{D\varphi(x_o)} Q \ o^* \ 1 \right) - \lambda I_N \right)
\]
since

\[P^* P = I_N, \quad P \left( \begin{array}{ccc} O & o^* & o \\ o^* & 1 & \end{array} \right) P^* = f_N \otimes f_N.\]

(3.31) completes the proof since \(f_i. f_N \rangle = 0\) if \(i \neq N\).

(Proof of Step II). Let \(x = (x_i)_{i=1}^{N-1} \in \mathbb{R}^{N-1}\) and \(\lambda \in \mathbb{R}\). Suppose that

\[
Q^* D \left( \frac{D\varphi(x_o)}{D\varphi(x_o)} \right) Q x = \lambda x.
\]

Then

\[
QQ^* D \left( \frac{D\varphi(x_o)}{D\varphi(x_o)} \right) \sum_{1 \leq i \leq N-1} x_i f_i = \lambda \sum_{1 \leq i \leq N-1} x_i f_i
\]
and henceforth by (3.31),

\[
D \left( \frac{D\varphi(x_o)}{D\varphi(x_o)} \right) \sum_{1 \leq i \leq N-1} x_i f_i = \lambda \sum_{1 \leq i \leq N-1} x_i f_i
\]

(3.34)
since, by (iii),

$$QQ^*(I_N - f_N \otimes f_N) = I_N - f_N \otimes f_N.$$  

It is easy to see that (3.34) implies (3.33).

Q. E. D.

For $i = 1, \cdots, N$, put

$$y_k(x) := \left( (\delta_{ij} - 1) \frac{\partial_j \varphi(x)}{|D\varphi(x)|} + \delta_{ij} \varphi(x) \right)_{j=1}^N.$$  

Then

**Lemma 8** Suppose that all eigenvalues of $D(D\varphi(x)\mid|D\varphi(x)|)$ are nonpositive. Then, for $i = 1, \cdots, N$,

$$\frac{\partial_i \varphi(x)}{|D\varphi(x)|} G(D\varphi(x), D^2\varphi(x)) = \det(Dy_k(x)).$$  

(Proof). For the sake of simplicity, we assume that $i = N$.

We first consider the case when $\partial_N \varphi(x) \neq 0$. By (ii) in Lemma 7, it is easy to see that the following holds:

$$\left( \begin{array}{c} I_{N-1} \\ -\frac{D\varphi(x)^*}{\partial_N \varphi(x)} \end{array} \right) Dg(x) = D\left( -\frac{D\varphi(x)}{|D\varphi(x)|} \right) + \left( \begin{array}{c} 0 \\ -D\varphi(x)^* \end{array} \right).$$  

(3.36)

By (i) and (iv) in Lemma 7, the eigenvalues and eigenvectors of $D(-D\varphi(x)\mid|D\varphi(x)|)$ on $\text{span}(g_1, \cdots, g_N)$ are real and are also those of the l.h.s. for short of (3.36).

We show that all eigenvalues of the l.h.s. of (3.36) are those of $D(-D\varphi(x)\mid|D\varphi(x)|)$ on $\text{span}(g_1, \cdots, g_N)$ and $-\partial_N \varphi(x)$.

By (i) and (iv) in Lemma 7, there exists an invariant subspace, which contains $f_N$, with an eigenvalue $\lambda$ of the l.h.s. of (3.36).

Take $\ell \geq 1$ such that

$$\left( \begin{array}{c} I_{N-1} \\ -\frac{D\varphi(x)^*}{\partial_N \varphi(x)} \end{array} \right) Dg(x) - \lambda \right)^\ell f_N = 0.$$  

(3.37)

Then $(-\partial_N \varphi(x) - \lambda)^\ell f_N \in \text{span}(g_1, \cdots, g_N)$ since

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\[ f_N = \frac{|D\varphi(x_o)|}{\partial_N \varphi(x_o)}((\delta_{jN})_{j=1}^N - g_N). \]

Hence \( \lambda = -\partial_N \varphi(x_o) \) by (i) in Lemma 7.

Suppose that \( \partial_N \varphi(x_o) = 0 \). Then, by (3.31) and (i) in Lemma 7, for \( x \in \mathbb{R}^N \),

\[ < f_N, D\gamma_N(x_o)x > = \left< f_N, D\left( \frac{D\varphi(x_o)}{|D\varphi(x_o)|} \right)x \right> = 0, \quad (3.38) \]

which implies that \( D\gamma_N(x_o)(\mathbb{R}^N) \) is at most (N-1)-dimensional and henceforth (3.35) holds.

Q. E. D.
4 Proof

In this section we prove the results in section 2.
(Proof of Theorem 1). By Lemmas 1-6, there exists a unique (nonrandom) solution \( \{D(t)\}_{0 \leq t < V_0} \) (see (3.6) for notation) of (1.13)-(1.14) on \([0, V_0)\) such that \( I_D(\cdot) \in C([0, V_0) : \mathcal{S}) \) and that the following holds: for any \( T \in [0, V_0) \) and \( \gamma > 0 \),

\[
\lim_{m \to \infty} P \left( \sup_{0 \leq t \leq T} d_S(Y_m(t, \cdot), I_D(t)(\cdot)) \geq \gamma \right) = 0. \tag{4.1}
\]

Therefore

\[
\lim_{m \to \infty} P \left( \sup_{0 \leq t \leq T} \|Y_m(t, \cdot) - I_D(t)(\cdot)\|_{L^2([-K,K]^N)} \geq \gamma \right) = 0, \tag{4.2}
\]

since, for \( m \geq 1 \) and \( t \in [0, T] \),

\[
\|Y_m(t, \cdot) - I_D(t)(\cdot)\|_{L^2([-K,K]^N)}^2 = \int_{[-K,K]^N} (Y_m(t,x) - 2Y_m(t,x)I_D(t)(x) + I_D(t)(x))^2dx.
\]

We prove that the following holds:

\[
\lim_{m \to \infty} P \left( \sup_{0 \leq t \leq T} \|X_m(t, \cdot) - I_D(t)(\cdot)\|_{L^2([-K,K]^N)} \geq \gamma \right) = 0. \tag{4.3}
\]

For any \( s \) and \( t \) for which \( 0 \leq s < t \leq T \),

\[
\|X_m(t, \cdot) - I_D(t)(\cdot)\|_{L^2([-K,K]^N)} \leq \|X_m(t, \cdot) - X_m(s, \cdot)\|_{L^2([-K,K]^N)} + \|X_m(s, \cdot) - Y_m(s, \cdot)\|_{L^2([-K,K]^N)} + \|Y_m(s, \cdot) - I_D(s)(\cdot)\|_{L^2([-K,K]^N)} + \|I_D(s)(\cdot) - I_D(t)(\cdot)\|_{L^2([-K,K]^N)}. \tag{4.4}
\]

Let \( U_{N^{1/2}/m}(D) := \{ x \in D| \text{dist}(x, D^c) > N^{1/2}/m \} \). Then

\[
\|X_m(t, \cdot) - X_m(s, \cdot)\|_{L^2([-K,K]^N)}^2 = \|X_m(t, \cdot) - X_m(s, \cdot)\|_{L^1([-K,K]^N)}^2 \geq 2N \sum_{z \in D_m} (Y_m(s, z) - Y_m(t, z)) \frac{1}{m} + \text{Vol}(D \setminus U_{N^{1/2}/m}(D)) \tag{4.5}
\]

\[
\leq 2N \sum_{z \in D_m} (Y_m(s, z) - Y_m(t, z)) \frac{1}{m} + \text{Vol}(D \setminus U_{N^{1/2}/m}(D))
\]
(see (2.2) for notation). Indeed, if \( x = (x_i)_{i=1}^{N} \in U_{-N^{1/2}/m}(D) \backslash (\cap Y_m(t, \cdot)^{-1}(1)) \), then \( Y_m(t, z) = 0 \) for some \( z = (z_i)_{i=1}^{N} \in \mathbb{Z}^N/m \) for which \( |x_i - z_i| \leq 1/m \) for all \( i = 1, \ldots, N \).

In the same way as in (3.5), by (4.5), for any \( \gamma > 0 \), there exists \( \delta > 0 \) such that the following holds: for any \( s \in [0, T - \delta] \),

\[
\lim_{m \to \infty} P( \sup_{s \leq s_1 \leq s + \delta} \| X_m(s_1, \cdot) - X_m(s, \cdot) \|_{L^2([-K, K]^N)} \geq \gamma ) = 0. \tag{4.6}
\]

Since, for any \( t \in [0, V_0] \), any subsequence of \( \{ C_m(t) \}_{m \geq 1} \) has a convergent subsequence (see the the proof of Lemma 5),

\[
\lim_{m \to \infty} \| Y_m(t, \cdot) - X_m(t, \cdot) \|_{L^2([-K, K]^N)} = 0 \tag{4.7}
\]
for all \( t \in [0, V_0] \), \( P_1 \)-almost surely (see the discussion after (3.18)). Hence, for any \( \gamma > 0 \),

\[
\lim_{m \to \infty} P( \| Y_m(s, \cdot) - X_m(s, \cdot) \|_{L^2([-K, K]^N)} \geq \gamma ) = 0. \tag{4.8}
\]

\( I_{D(\cdot)} \in \mathcal{C}([0, V_0] : L^2([-K, K]^N)) \), since

\[
\| I_{D(s)}(\cdot) - I_{D(t)}(\cdot) \|_{L^2([-K, K]^N)}^2 = \int_{[-K, K]^N} I_{D(s)}(x) dx - \int_{[-K, K]^N} I_{D(t)}(x) dx,
\]
and since \( t \mapsto \int_{[-K, K]^N} I_{D(t)}(x) dx \) is continuous on \([0, V_0]\).

(4.2) and the discussion after (4.3) show that (4.3) is true.

Recall Lemmas 2-3 and the notations therein. For \( T < V_0 \), take \( x_0 \in D(T) \) and \( r_0 \) so that \( U_{3r_0}(x_0) \subset D(T) \). For sufficiently large \( k \geq 1 \),

\[
U_{3r_0}(x_0) \subset (\cap Y_m(T, \cdot)^{-1}(1))^o \cap D, \quad P_1 - a.s.,
\]

since

\[
\lim_{k \to \infty} \| X_m(T, \cdot) - I_{D(T)}(\cdot) \|_{L^2([-K, K]^N)} = 0, \quad P_1 - a.s.
\]
by Lemma 2 and (4.7) (see the discussion below (4.2)). Hence in the same way as in Lemma 3,
\[
V_0 \leq \liminf_{k \to \infty} \sum_{z \in D_{m_k}} \left( Y_{m_k}(T, z) - Y_{m_k}(\tau_{m_k}, z) \right) \frac{1}{m_k^{n}} \tag{4.9}
\]

which implies that \(4.3\) holds for \(T < 2V_0\). Repeating the same procedure as above and then letting \(r_0 \downarrow 0\), \(4.3\) holds for all \(T < T^* := \text{Vol}(D)\).

Put

\[
D(t) = \emptyset \quad \text{for } t \geq T^*. \tag{4.10}
\]

Then \(I_{D(t)} \in C([0, \infty) : L^2([-K, K]^N))\) and \(\{D(t)\}_{t \geq 0}\) is a unique solution to (1.13)-(1.14) on \([0, \infty)\) by Lemma 6, since \(t \mapsto I_{D(t)}\) is nonincreasing and since

\[
\text{Vol}(D(t)) = \text{Vol}(D(0)) - t \downarrow 0, \quad \text{as } t \uparrow T^*, \tag{4.11}
\]

by (1.14).

We prove (2.7). Take a sufficiently small positive \(\varepsilon\) so that

\[
\text{Vol}(D(t)) \leq \left( \frac{\gamma}{4} \right)^2 \quad \text{for } t \geq t_\varepsilon := T^* - \varepsilon. \tag{4.12}
\]

Then

\[
P(\sup_{t \geq 0} \|X_{m}(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \geq \gamma) \tag{4.13}
\]

\[
\leq P(\sup_{0 \leq t \leq t_\varepsilon} \|X_{m}(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \geq \gamma) + P(\sup_{t \geq t_\varepsilon} \|X_{m}(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \geq \gamma)
\]

\[
\leq 2P(\sup_{0 \leq t \leq t_\varepsilon} \|X_{m}(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \geq \frac{\gamma}{2}) \to 0 \quad \text{as } m \to \infty
\]

since for \(t \geq t_\varepsilon\),

\[
\|X_{m}(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)}
\]

\[
\leq \|X_{m}(t_\varepsilon, \cdot)\|_{L^2([-K, K]^N)} + \|I_{D(t_\varepsilon)}(\cdot)\|_{L^2([-K, K]^N)} + 2\|I_{D(t_\varepsilon)}(\cdot)\|_{L^2([-K, K]^N)}.
\]

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(Proof of Corollary 1). Since $D$ is convex,

$$\text{co } Y_m(t, \cdot)^{-1}(1)^\circ \cap D = (\text{co } Y_m(t, \cdot)^{-1}(1))^\circ =: D_m(t).$$

For $T < T^*$, take $x_0 \in D(T)$ and $r_0$ so that $U_{r_0}(x_0) \subset D(T)$ (see (3.6) for notation). Then, for sufficiently large $m$, $U_{3m}(x_0) \subset D_m(0)$. Consider cones

$$\text{cone}(x) := \text{co } (\{x\} \cup U_0^-) \quad (x \in D^-),$$

and for $r > 0$, put

$$V(r) := \inf_{x \in \partial D} \text{Vol} \left( \text{cone}(x) \cap H(x_0 - x, x + r(x_0 - x)) \right), \quad (4.14)$$

$$V_m(r) := \inf_{x \in \partial D_m(0)} \text{Vol} \left( \text{cone}(x) \cap H(x_0 - x, x + r(x_0 - x)) \right), \quad (4.15)$$

Then for $\gamma > 0$ and sufficiently large $m \geq 1$, by Theorem 1,

$$P(\sup_{0 \leq t \leq T} d_H(D_m(t), D(t)) \geq \gamma) \leq P(\|I_{D_m(T)}(\cdot) - I_{D(T)}(\cdot)\|_{L^2([-K,K])} \geq V_0) + P(U_0 \subset D_m(T), \sup_{0 \leq t \leq T} d_H(D_m(t), D(t)) \geq \gamma) \to 0 \quad (as \ m \to \infty).$$

Indeed,

$$P(U_0 \subset D_m(T), \sup_{0 \leq t \leq T} d_H(D_m(t), D(t)) \geq \gamma) \leq P(\sup_{0 \leq t \leq T} \|I_{D_m(t)}(\cdot) - I_{D(t)}(\cdot)\|_{L^2([-K,K])} \geq \min(V(\gamma), V_m(\gamma))),$$

and $V_m(\gamma) \geq V(\gamma)$ for all $m \geq 1$. Q. E. D.

(Proof of Corollary 2). For $r \in \mathbb{R}$, let $\{D_r(t)\}_{t \geq 0}$ denote a unique solution of (1.13)-(1.14) with $D_r(0) = h^{-1}((r, \infty))$ on $[0, \infty)$. Notice that
\[ D_r(\cdot) := \begin{cases} \mathbb{R}^N & \text{if } r < \inf \{ h(x) | x \in \mathbb{R}^N \}, \\ \emptyset & \text{if } r \geq \sup \{ h(x) | x \in \mathbb{R}^N \}. \end{cases} \]  

(4.17)

Put

\[ u(t, x) := \sup \{ r \in \mathbb{R} | x \in D_r(t) \}. \]  

(4.18)

Then, for all \( t \geq 0 \) and \( r \in \mathbb{R} \) for which \( D_r(t) \neq \emptyset, \mathbb{R}^N \),

\[ u(t, \cdot)^{-1}((r, \infty)) = D_r(t), \]  

(4.19)

since \( D_r(t) = D_{r^+}(t) := \cup_{r^+} D_r(t) \) by (1.13).

Indeed, \( D_r(0) = D_{r^+}(0) \); and if \( \bar{r} - r \) is positive and is sufficiently small, then \( D_{r}(t) \neq \emptyset \) by (b) in Theorem 2, and

\[ \int_{\mathbb{R}^N} (I_{D_{r^+}}(x) - I_{D_r}(x)) dx = \int_{\mathbb{R}^N} (I_{D_{\bar{r}}} - I_{D_r}(x)) dx. \]  

By Lemma 6 and (4.19), \( u \) is continuous.

For \( m \geq 1 \), put

\[ \begin{aligned} k_{m,1} & := m \sup \{ h(y) | y \in \mathbb{R}^N \}, \\ k_{m,0} & := m \inf \{ h(y) | y \in \mathbb{R}^N \} - 1. \end{aligned} \]

Then

\[ \sum_{k_{m,0} \leq k \leq k_{m,1}} \frac{k}{m} (I_{D^{k,m}} - I_{D^{k+1,m}}(x)) \]  

(4.20)

\[ = \sum_{k_{m,0} \leq k \leq k_{m,1}} \frac{1}{m} I_{D^{k,m}}(x) - \frac{k_{m,1} + 1}{m} I_{D^{k+1,m}}(x) + \frac{k_{m,0}}{m} I_{D^{k,m,0}}(x). \]

Since \( I_{D^{k+1,m}}(x) \equiv 0 \) and since \( I_{D^{k,m,0}}(x) \equiv 1 \), the following holds:

for any \( \varphi \in C_0(\mathbb{R}^N) \) and any \( t \geq 0 \),

\[ \int_{\mathbb{R}^N} \varphi(x) \sum_{k_{m,0} \leq k \leq k_{m,1}} \frac{k}{m} (I_{D^{k,m}}(x) - I_{D^{k+1,m}}(x)) \]  

(4.21)

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\[- \sum_{k_m, o < k < k_m, 1} \frac{k}{m} (I_{D_{\frac{k}{m}}(t)}(x) - I_{D_{\frac{k+1}{m}}(t)}(x)]dx \]
\[= \int_0^t \sum_{k_m, o < k < k_m, 1} \frac{1}{m} \int_{\mathbb{R}^N} \varphi(x) \omega_1 (I_{D_{\frac{k}{m}}(s)}(\cdot), dx). \]

Letting \(m \rightarrow \infty\) in (4.21), one can show that \(u\) is a solution to (1.15) by Lemma 4, since \(\text{co } D_{\frac{m+1}{m}}(s) \rightarrow \text{co } D_r(s)\) as \(m \rightarrow \infty\) for \(r \in [\text{inf}\{u(s, y)|y \in \mathbb{R}^N\}, \text{sup}\{u(s, y)|y \in \mathbb{R}^N\}])\), provided \(D_r(s) \neq \emptyset, \mathbb{R}^N\).

Let \(v \in C([0, \infty) \times \mathbb{R}^N)\) be a solution to (1.15) with \(v(0, \cdot) = h(\cdot)\). Then for \(n \geq 1, r \in [\text{inf}\{h(y)|y \in \mathbb{R}^N\}, \text{sup}\{h(y)|y \in \mathbb{R}^N\}]), \) provided \(\varphi \in C_o(\mathbb{R}^N)\) and \(t \geq 0, \)

\[\int_{\mathbb{R}^N} \varphi(x)\{\eta_n(v(0, x) - r) - \eta_n(v(t, x) - r)\}dx \quad (4.22)\]
\[= \int_0^t ds \int_{\mathbb{R}} \frac{d\eta_n(\bar{\tau} - r)}{d\bar{\tau}} \int_{\mathbb{R}^N} \varphi(x) \omega_\tau(v(s, \cdot), dx) \]

(see (3.24) for notation). Let \(n \rightarrow \infty\) in (4.22). Then we see that \(\tilde{D}_r(t) := v(t, \cdot)^{-1}((r, \infty))\) is a solution to (1.14) on \([0, \infty)\) by Lemma 4 and the continuity of \(v\).

We prove that \(v(t, \cdot)^{-1}((r, \infty))\) satisfies (1.13). For \(x \in (\text{co } \tilde{D}_r(t)) \cap \tilde{D}_r(0), \)

\[
\int_{\mathbb{R}^N} \varphi(x)\{I_{D_r(0)} - I_{D_r(t)}\}dx = \int_0^t ds \int_{\mathbb{R}^N} \varphi(x) \omega_1 (I_{D_r(s)}(\cdot), dx) = 0, \quad (4.23)\]

which implies that \(x \in (U_\delta(x) \subset \tilde{D}_r(t)). \) Hence (1.13) holds.

The uniqueness of \(u\) follows from that of \(D_r(\cdot)\) for all \(r\).

Q. E. D.

Theorem 2 is an easy consequence of Theorem 1 and Lemma 6 and we omit the proof.

(Proof of Theorem 3).

(Step 1). We first show that \(u(t, x) := I_{D(t)}(x)\) is a viscosity supersolution of (1.20) in \((0, \infty) \times \mathbb{R}^N\).

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Let \( \psi \in \mathcal{A}((0, \infty) \times \mathbb{R}^N) \) and assume that \( u - \psi \) attains a local minimum at \( (t_0, x_0) \in (0, \infty) \times \mathbb{R}^N \). Without loss of generality, we may assume that \( u(t_0, x_0) = \psi(t_0, x_0) \) and that \( u(t, x) > \psi(t, x) \) for all \( (t, x) \in (0, \infty) \times \mathbb{R}^N \setminus \{(t_0, x_0)\} \) (see [8]).

If \( x_0 \not\in \partial(co \ D(t_0)) \cap \partial D(t_0) \), then \( \partial_t \psi(t_0, x_0) \geq 0 \).

Indeed, \( t \mapsto u(t, x_0) \) is constant if \( t_0 - t \) is a sufficiently small positive number, from which \( \psi(t_0, x_0) > \psi(t, x_0) \) for such \( t \).

Suppose that \( x_0 \in \partial(co \ D(t_0)) \cap \partial D(t_0) \). Then \( u(t_0, x_0) = 0 \), and \( D\psi(t_0, x_0) = 0 \) or \( \sigma^+(u, D\psi(t_0, x_0), t_0, x_0) = 1 \).

Indeed, if \( D\psi(t_0, x_0) \neq 0 \), then for \( y \) for which \( y + x_0 \not\in H(D\psi(t_0, x_0), x_0) \) and for \( r > 0 \), by the mean value theorem, there exists \( \theta \in (0, 1) \) such that

\[
    u(t_0, x_0 + ry) > \psi(t_0, x_0 + ry) = \psi(t_0, x_0) + r < D\psi(t_0, x_0 + \theta ry), y >> 0,
\]

provided \( r \) is sufficiently small, by the continuity of \( D\psi \).

(Case 1). We first consider the case when \( D\psi(t_0, x_0) = 0 \). We may assume that there exist \( f \in \mathcal{F} \) and \( \varphi_1 \in C^2((0, \infty)) \) such that

\[
    \psi(t, x) = -f(|x - x_0|) - \varphi_1(t) \tag{4.24}
\]

(see [21]).

For \( A > 0 \) and \( m \geq 2 \), put

\[
    \psi_{m,A}(t, x) = \psi(t, x) - A\{|t - t_0|^2 + |x - x_0|^m\}. \tag{4.25}
\]

Then

\[
    \partial_t \psi_{m,A}(t_0, x_0) = \partial_t \psi(t_0, x_0), \quad D\psi_{m,A}(t_0, x_0) = D\psi(t_0, x_0), \tag{4.26}
\]

and

\[
    U^+_{m,A,\varepsilon} := \{(t, x) \in (0, \infty) \times \mathbb{R}^N | \psi_{m,A}(t, x) + \varepsilon > u(t, x)\} \tag{4.27}
\]

\[\subset U_{(2\varepsilon/A)^{1/m}}((t_0, x_0))\]

(\( \varepsilon \in (0, A) \)), and the following holds: for \( t \geq 0 \)

\[
    \lim_{x \to x_0} G(D\psi_{N,A}(t, x), D^2\psi_{N,A}(t, x)) = NA. \tag{4.28}
\]
We argue by contradiction. We consider \( \psi_{N,A} \) instead of \( \psi \). When it is not confusing, we omit \( N,A \) for the sake of simplicity.

Assume that the following holds:

\[
\partial_t \psi(t_0, x_0) < 0.
\] (4.29)

By reselecting \( A > 0 \) sufficiently small and \( \varepsilon > 0 \) sufficiently small compared to \( A \) if necessary, we may assume that

\[
\partial_t \psi(t, x) + R\left( \frac{D\psi(t, x)}{|D\psi(t, x)|} \right) G(D\psi(t, x), D^2\psi(t, x)) + \varepsilon < 0 \quad \text{on} \ U^+_\varepsilon, \quad (4.30)
\]

and that

\[
U^+_\varepsilon = \bigcup_{t>0} \{ t \} \times (\psi(t, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(t)^c). \quad (4.31)
\]

We may also assume that \( x \mapsto \psi(t, x) \) is strictly concave on \( U^+_\varepsilon \) and henceforth \( x \mapsto (\psi(s, x), D\psi(s, x)/|D\psi(s, x)|) \) is one-to-one on some neighborhood of \( \partial\psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c \), provided \( \psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c \neq \emptyset \).

Indeed, if \( \psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c \neq \emptyset \), then \( -\varepsilon \) is not the maximum of \( \psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c \) and hence \( D\psi(s, \cdot) \neq 0 \) on some neighborhood of \( \partial\psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c \).

For \( t \geq 0 \),

\[
\int_{\mathbb{R}^N} (\zeta_k(\eta_m(\psi(t, x) + \varepsilon) - u(t, x))
- \zeta_k(\eta_m(\psi(0, x) + \varepsilon) - u(0, x))) \, dx
= \int_{\mathbb{R}^N} dx \int_0^t (-\zeta_k(\eta_m(\psi(s, x) + \varepsilon) - u(s, x))u(ds, x)
+ \eta_k(\eta_m(\psi(s, x) + \varepsilon) - u(s, x)) \frac{d\eta_m(\psi(s, x) + \varepsilon)}{dr} \partial_s\psi(s, x) \, ds)
\] (see (3.24)-(3.25) for notation).

Letting \( k \to \infty \) in (4.32), by (4.31),

\[
0 \leq \int_0^t ds \{ \int_{\psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c} \eta_m(\psi(s, x) + \varepsilon) \omega_1(u(s, \cdot), dx)
+ \int_{\psi(s, \cdot)^{-1}((-\varepsilon, -\varepsilon+1/m)) \cap D(s)^c} \frac{d\eta_m(\psi(s, x) + \varepsilon)}{dr} \partial_s\psi(s, x) \, dx \}
\] (4.33)
For $s$ for which $\psi(s, \cdot)^{-1}((-\varepsilon, -\varepsilon + 1/m)) \cap D(s)^c \neq \emptyset$ and sufficiently large $m \geq 1$, by Lemma 8,

\[
\int_{\psi(s, \cdot)^{-1}((-\varepsilon, -\varepsilon + 1/m)) \cap D(s)^c} \frac{d\eta_m(\psi(s, x) + \varepsilon)}{dr} \partial_s \psi(s, x) dx \quad (4.34)
\]

\[
< - \int_{\varepsilon}^{\varepsilon+1/m} d\eta_m(r + \varepsilon) dr \int_{\partial \psi(s, r)} \frac{D\psi(s, x)}{D\psi(s, x)} (R(p) + \varepsilon \sup \{G(D\psi(s, x), D^2\psi(s, x)) : (s, x) \in U_\varepsilon^+ \}^{-1}) d\mathcal{H}^{N-1}(p)
\]

\[
\rightarrow - \int_{\cup, r > -\varepsilon} \left\{ \frac{D\psi(s, x)}{D\psi(s, x)} : x \in \partial \psi(s, r)^{-1}((r, \infty)) \cap D(s)^c \right\}
\]

(4.33)-(4.34) contradicts to

\[
\{ p \in S^{N-1} : \sigma^+(u, -p, s, x) = 1 \text{ for some } x \in \psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c \}
\]

\[
\subset \cup_r > -\varepsilon \left\{ \frac{D\psi(s, x)}{D\psi(s, x)} : x \in \partial \psi(s, r)^{-1}((r, \infty)) \cap D(s)^c \right\}
\]

since

\[
\eta_m(\psi(s, x) + \varepsilon) \rightarrow 1 \quad \text{if } x \in \psi(s, \cdot)^{-1}((-\varepsilon, \infty)), \text{ as } m \rightarrow \infty.
\]

(Case 2). Next we consider the case when $\sigma^+(u, D\psi(t_0, x_0), t_0, x_0) = 1$. By (ii)-(iv) in Lemma 7, all eigenvalues of $-D(D\psi(t_0, x_0)/|D\psi(t_0, x_0)|)$ are nonnegative since the function $x \mapsto \psi(t_0, x)$ takes a maximum $\psi(t_0, x)$ on the set $\{x_0 + y \in \mathbb{R}^N | < y, D\psi(t_0, x_0) >= 0 \}$.

For $A > 0$, all eigenvalues of $-D(D\psi_{2,A}(t_0, x_0)/|D\psi_{2,A}(t_0, x_0)|)$ as a mapping on the set $\{y \in \mathbb{R}^N | < y, D\psi_{2,A}(t_0, x_0) >= 0 \}$ are greater than or equal to $2A/|D\psi(t_0, x_0)|$ (see (3.31)-(3.32)) since, in Lemma 7, 1 and $f_1, \cdots, f_{N-1}$ are a eigenvalue and eigenvectors of $(g_1 \cdots g_N)$, respectively.

We argue by contradiction. Assume that the following holds:

\[
\partial_1 \psi(t_0, x_0) + R \left( \frac{D\psi(t_0, x_0)}{|D\psi(t_0, x_0)|} \right) G(D\psi(t_0, x_0), D^2\psi(t_0, x_0)) < 0. \quad (4.35)
\]

We consider $\psi_{2,A}$ instead of $\psi$. When it is not confusing, we omit $2,A$ for the sake of simplicity. By reselecting $A, \varepsilon > 0$ if necessary, we may assume that (4.30)-(4.31) hold.
One can also assume, in $U_{(2\varepsilon/A)^{1/2}}((t_0, x_0))$, that $\partial_t \psi(s, x) \neq 0$ and all eigenvalues of $-D(D\psi(s, x)/|D\psi(s, x)|)$ as a mapping on the set $\{ y \in \mathbb{R}^N \mid < y, D\psi(s, x) >= 0 \}$ are greater than or equal to $A/|D\psi(t_0, x_0)|$, and $x \mapsto \gamma_i(s, x)$ is one-to-one for some $i \in \{1, \ldots, N\}$ by the inverse function theorem and (v) in Lemma 7, and Lemma 8.

In the same way as in (4.32)-(4.34), we obtain a contradiction.

(Step II). We show that $u^-(t, x) = I_{D(t)^-}(x)$ is a viscosity subsolution of (1.20).

Let $\psi \in \mathcal{A}((0, \infty) \times \mathbb{R}^d; \mathbb{R}^d)$ and assume that $u^--\psi$ attains a maximum at $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^d$. We may assume as well that $u^-(t_0, x_0) = \psi(t_0, x_0)$, so that $u^-(t, x) < \psi(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbb{R}^d \setminus \{(t_0, x_0)\}$ (see [8]).

Since $t \mapsto u^-(t, x)$ is nonincreasing, $\partial_t \psi(t_0, x_0) \leq 0$.

Hence we only have to consider the case when the following holds: $D\psi(t_0, x_0) \neq 0$, and

$$
\sigma^-(u^-, D\psi(t_0, x_0), t_0, x_0) = 1, \quad R\left(\frac{D\psi(t_0, x_0)}{|D\psi(t_0, x_0)|}\right)G(D\psi(t_0, x_0), D^2\psi(t_0, x_0)) > 0.
$$

In particular, $u^-(t_0, x_0) = 1$. By adding to $\psi$ the function $(t, x) \mapsto A\{t - s|2 + |x - y|^2\}$, with a sufficiently small $A > 0$, if necessary, we may assume that

$$
U^-_\varepsilon := \{(t, x) \in (0, \infty) \times \mathbb{R}^d \mid \psi(t, x) - \varepsilon < u^-(t, x)\} \quad (\varepsilon > 0) \quad (4.36)
$$

is contained in the set $U_{(\varepsilon/A)^{1/2}}((t_0, x_0))$.

We argue by contradiction. Assume that the following holds:

$$
\partial_t \psi(t_0, x_0) + R\left(\frac{D\psi(t_0, x_0)}{|D\psi(t_0, x_0)|}\right)G(D\psi(t_0, x_0), D^2\psi(t_0, x_0)) > 0. \quad (4.37)
$$

By reselecting $\varepsilon > 0$ if necessary, we may assume that

$$
\partial_t \psi(t, x) + R\left(\frac{D\psi(t, x)}{|D\psi(t, x)|}\right)G(D\psi(t, x), D^2\psi(t, x)) - \varepsilon > 0, \quad (4.38)
$$

and $u^-(t, x) = 1$ on $U^-_\varepsilon$ by the continuity of $\psi$. 

\[35\]
Put $\tilde{\eta}_m(r) = \eta_m(r + 1/m)$ for $r \in \mathbb{R}$ and $m \geq 1$. In the same way as in (Step 1), considering $u^-(t, x) = \tilde{\eta}_m(\psi(t, x) - 1 - \varepsilon) \leq \eta_m(\psi(t, x) + \varepsilon) - u(t, x)$, we obtain a contradiction.

Q. E. D.

(Proof of Corollary 3).
We first show that $u$ is a viscosity supersolution of (1.20) in $(0, \infty) \times \mathbb{R}^N$. Let $\varphi \in \mathcal{A}((0, \infty) \times \mathbb{R}^N)$ and assume that $u - \varphi$ attains a minimum at $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^N$. We may assume that $u(t_0, x_0) = \varphi(t_0, x_0)$, so that $u(t, x) > \varphi(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbb{R}^N \setminus \{(t_0, x_0)\}$ (see [8]). By subtracting a constant, we may assume that $\varphi \leq u < 0$.

Put $r_0 := \varphi(t_0, x_0)$ and

$$u_r(t, x) := I_{u^{-1}(t, y)(r, 0)}(x) \quad (r < 0),$$

then

$$u_{r_0}(t, x) \geq \frac{\varphi(t, x)}{|r_0|} + 1 \quad \text{for all } (t, x) \in (0, \infty) \times \mathbb{R}^N,$$

(4.40)

where the equality holds if and only if $(t, x) = (t_0, x_0)$.

Since $u_r$ is a viscosity supersolution of (1.20) in $(0, \infty) \times \mathbb{R}^N$ by Corollary 2 and Theorem 3, and since

$$\sigma^+(u_{r_0}, D(\varphi(t_0, x_0)/|r_0| + 1), t_0, x_0) = \sigma^+(u, D\varphi(t_0, x_0), t_0, x_0),$$

(1.25) holds.

Next we show that $u$ is a viscosity subsolution of (1.20) in $(0, \infty) \times \mathbb{R}^N$.
Let $\varphi \in \mathcal{A}((0, \infty) \times \mathbb{R}^d)$ and assume that $u - \varphi$ attains a maximum at $(t_1, x_1) \in (0, \infty) \times \mathbb{R}^d$. We may assume as well that $u(t_1, x_1) = \varphi(t_1, x_1)$, so that $u(t, x) < \varphi(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbb{R}^d \setminus \{(t_1, x_1)\}$ (see [8]).

By adding a constant, we may assume that $\varphi \geq u > 0$.

Put $r_1 := \varphi(t_1, x_1)$ and

$$u_r^-(t, x) := I_{u^{-1}((t, y)(r, \infty))}(x).$$

Then

$$u_r^-(t, x) \leq \frac{\varphi(t, x)}{r_1} \quad \text{for all } (t, x) \in (0, \infty) \times \mathbb{R}^N,$$
where the equality holds if and only if \((t, x) = (t_1, x_1)\).

Since \(u_{r_i}^-\) is a viscosity subsolution of (1.20) in \((0, \infty) \times \mathbb{R}^N\) by Corollary 2 and Theorem 3, and since

\[
\sigma^-(u_{r_i}^-, D(\varphi(t_1, x_1)/r_1), t_1, x_1) = \sigma^- (u, D\varphi(t_1, x_1), t_1, x_1),
\]

(1.27) holds.

Q. E. D.
References


